Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Professor M. Cantor, who was presiding over the section, seemed perturbed, perhaps at the very idea of future historians of mathematics, and strenuously insisted that the idea of positional notation for number and the symbolization of zero should be accredited to the Babylonians. Of course we know that there are some fragmentary matters which point to the Babylonians using sixty as a number radix, but I have never learned that any Babylonian symbol for zero has been discovered, and the question to which I called attention remained untouched. But now, if we may accept the testimony of Y. Mikami, of Tokio, one of those questions has been answered. He says, "I have found very important relations between the mathematics of India and of China." But we know that Japanese hieroglyphic writing, like Japanese Buddhism, came through China.

Therefore the appearance of our symbol for zero in Hindustan and Japan is not one of the mysterious and seemingly inexplicable coincidences, such as the prehistoric appearance and importance of the cross in the pre-Columbian religions of Yucatan and Mexico. For here we find a water-way connecting Hindustan and Japan through China. But the source of the flow still remains undetermined.

Use of the abacus might have been what suggested the symbolization of zero. But Hindustan from time immemorial used systems of abacal calculation. Greece, and even mathematically stupid Rome, had the abacus.

Every claim for importance in the world of ideas hitherto made for China has evaporated into nothingness. They did not even invent gunpowder or the mariner's compass. Remember their exaltation and subsequent,downfall in the history of astronomy.

So it will be as to their claim to our zero symbol. They got it, as they did their Buddhism, from India, and passed on both to Japan.

ANNAPOLIS, MD., *March 7, 1903.*

HARMONIC PAIRS IN THE COMPLEX PLANE.

A PURELY GEOMETRICAL TREATMENT FOR CERTAIN MAPS DEFINED BY THE SUBSTITUTION $w=\frac{1}{2}(z+\frac{1}{z})$.

By ARCHIBALD HENDERSON. Ph. D.. Associate Professor of Mathematics,
University of North Carolina. Chapel Hill.

§1. A subject of especial interest in the Theory of Functions is the conception of harmonic pairs in the complex plane. The proof given below offers initially a direct interpretation of the equation

$$\left|\frac{z-z_1}{z-z_2}\right|=\rho\ldots.(1),$$

and leads subsequently to the interpretation of the harmonic relation

$$\frac{z_1'-z_1}{z_2-z_1'} \div \frac{z_2'-z_1}{z_2-z_2'} = -1 \ldots. (2),$$

where z, z_1, z_2, z_1', z_2' are complex quantities and ρ is a constant.

In a well-known text-book* the method of interpreting equation (1) seems essentially artificial, although the logic is entirely sound. Instead of interpreting the equation directly, which is the problem set, the writer employs a property of the configuration, wholly unsuspected by one who reads the subject for the first time, which leads, after a series of transformations, to the equation in question.

For the following proof, several lemmas will be introduced.

Lemma 1. If $\frac{z-z_1}{z_2-z}=\kappa$, where κ is real, then z, z_1, and z_2 are collinear. Writing the equation in the form

$$z=\frac{z_1+\kappa z_2}{1+\kappa}$$

we may say that z divides the stroke from z_1 to z_2 in the ratio $+k : 1$.†

Lemma 2. If $\operatorname{am}\frac{z-z_1}{z-z_2}=\alpha$, then z moves on the arc of a circle through the points z_1 and z_2.‡

Lemma 3. If $\frac{z_1'-z_1}{z_2-z_1'} \div \frac{z_2'-z_1}{z_2-z_2'}=\lambda$, where λ is a real quantity, then the four points z_1, z_2, z_1', and z_2' are concyclic.§

This lemma is an immediate consequence of Lemma 2.

Consider now the equation (1) above and its interpretation. It is satisfied *in particular* by the two equations

$$\frac{z-z_1}{z-z_2}=\rho, \quad \frac{z-z_1}{z-z_2}=-\rho,$$

shown by taking absolute values on both sides of these two equations. Hence the locus represented by equation (1) crosses the line through the two points z_1 and z_2 at the two points

$$z_\mu=\frac{z_1+\rho z_2}{1+\rho}; \quad z_\nu=\frac{z_1-\rho z_2}{1-\rho},$$

---

* Harkness and Morley's *Introduction to Analytic Functions.*

† Harkness and Morley: *Introduction to Analytic Functions*, §18.

‡ Ibidem, §21.

§ Burkhardt: *Functionen Theorie*, Erster Band. §15, VI.
Harkness and Morley: *Introduction to Analytic Functions*, §25.

which follows from Lemma 2. It is to be noted that since the points z_μ and z_ν divide the stroke from z_1 to z_2 both externally and internally in the same ratio, the pairs z_1 and z_2, z_μ and z_ν are harmonic.

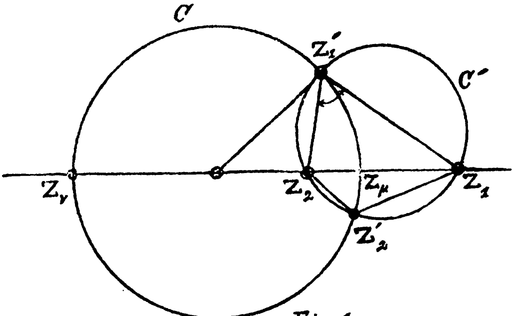

Fig. 1.

Also $z_c=\frac{1}{2}(z_\mu+z_\nu)$

$$=\frac{1}{2}\left(\frac{z_1+\rho z_2}{1+\rho}+\frac{z_1-\rho z_2}{1-\rho}\right)$$

$$=\frac{z_1-\rho^2 z_2}{1-\rho^2}$$

and this point divides the segment z_1 to z_2 externally in the ratio $1:\rho^2$.

Now $\left|\frac{z-z_1}{z-z_2}\right|=\rho$ requires that $\frac{z-z_1}{z-z_2}=\rho(\cos\theta+i\sin\theta)$ and accordingly $z=\frac{z_1-z_2\rho(\cos\theta+i\sin\theta)}{1-\rho(\cos\theta+i\sin\theta)}$.

Forming now the difference $z-z_c$ we obtain after reduction

$$z-z_c=\frac{\rho(z_1-z_2)}{1-\rho^2}(\cos\theta+i\sin\theta)\left[\frac{1-\rho(\cos\theta-i\sin\theta)}{1-\rho(\cos\theta+i\sin\theta)}\right].$$

Taking absolute values on both sides of this equation and noting that

$$|\cos\theta+i\sin\theta|=1,$$
$$|1-\rho(\cos\theta+i\sin\theta)|=|1-\rho(\cos\theta-i\sin\theta)|$$

we obtain

$$|z-z_c|=\frac{\rho\,|z_1-z_2|}{1-\rho^2}=\text{constant},$$

and since z_c is a fixed point, we have the final result, which may be stated as follows:

The equation $\left|\frac{z-z_1}{z-z_2}\right|=\rho$ represents a circle C of radius $\frac{\rho\,|z_1-z_2|}{1-\rho^2}$ with its center on the line joining the points z_1 and z_2 and dividing the segment from z_1 to z_2 externally in the ratio $1:\rho^2$.

Now through the points z_1 and z_2 draw any circle C', cutting the circle C in the two points z_1' and z_2'. The equations of the two finite portions of the arc of this circle C', by Lemma 2, are

$$\left.\begin{aligned}\operatorname{am}\frac{z-z_1}{z-z_2}&=\alpha\\ \\ \operatorname{am}\frac{z-z_1}{z-z_2}&=\alpha-\pi\end{aligned}\right\}\ldots.(C').$$

That the circles C and C' are orthogonal can be shown in the following manner. We have the three equations

$$|z_1'-z_c|=\frac{\rho\,|z_1-z_2|}{1-\rho^2},$$

$$|z_1-z_c|=\left|z_1-\frac{z_1-\rho^2z_2}{1-\rho^2}\right|=\frac{-\rho^2}{1-\rho^2}\,|z_1-z_2|,$$

$$|z_2-z_c|=\left|z_2-\frac{z_1-\rho^2z_2}{1-\rho^2}\right|=\frac{-1}{1-\rho^2}\,|z_1-z_2|,$$

and therefore,

$$|z_1'-z_c|^2=|z_1-z_c|\times|z_2-z_c|.$$

That is, the join of the two points z_c and z_1' is a tangent to the circle and hence the circles C and C' are orthogonal.*

Since $\dfrac{|z_1'-z_c|}{|z_2-z_c|}=\dfrac{|z_1-z_c|}{|z_1'-z_c|}$, the triangles $z_cz_2z_1'$ and $z_cz_1'z_1$ are similar, having the angle $z_2z_cz_1'$ in common. Putting

$$|z_1-z_c|\times|z_2-z_c|=r^2,$$

the points z_1, z_2 are defined as *inverse* with respect to the circle of center z_c and radius r; and this circle will be said to be drawn *about* any such pair of points.

Another definition of inverse points, found in Harkness and Morley's *Introduction*, based on the orthogonality of the circles C and C', is as follows:

Two points z_1, z_2 are said to be inverse to a circle, when every circle through z_1, z_2 cuts the given circle orthogonally.

§2. The next problem is to interpret the geometric meaning of the equation

$$\frac{z_1'-z_1}{z_2-z_1'}\div\frac{z_2'-z_1}{z_2-z_2'}=-1.$$

Since -1 is a real quantity, the four points z_1, z_2, z_1', z_2' are concyclic (Lemma 3). Also clearing out and taking absolute values on both sides, we obtain

---

* The orthogonality of the two circles may also be found as follows: Isogonality is an invariant property from every linear substitution of the form $w=(az+b)/(cz+d)$. (Forsyth: *Theory of Functions*, §258, page 514). Making the substitution $w=(z-z_1)/(z-z_2)$ in the two equations $|(z-z_1)/(z-z_2)|=\rho$, $\operatorname{am}[(z-z_1)/(z-z^2)]=\alpha$, their maps are $|w|=\rho$, $w=\alpha$,—circle with the origin as center and a half-ray through the origin. Since these figures are orthogonal in the w-plane, their images in the z-plane are also orthogonal.

$$\frac{|z_1'-z_1|}{|z_1'-z_2|}=\frac{|z_2'-z_1|}{|z_2'-z_2|}$$

and hence the points z_1' and z_2' lie on the locus defined by the equation

$$\left|\frac{z-z_1}{z-z_2}\right|=\rho.$$

These four points z_1, z_2, z_1', and z_2' lie on the circle C', two of them, z_1' and z_2', are on C, while two of them, z_1 and z_2, are inverse with respect to C. The pairs z_1 and z_2, z_1' and z_2' are defined as *harmonic pairs* of points.

Since the straight line through z_1 and z_2 may be regarded as a circle of infinite radius, it follows that the pairs z_ν and z_μ, z_1 and z_2 are harmonic pairs on a line, a fact which was proved in a different manner at the beginning of §1.

II.

§3. A very suggestive problem, illustrative of the Riemann surface, is to find the images of certain configurations—straight lines through the origin and concentric circles with origin as center—defined by the equation of correspondence

$$w=\tfrac{1}{2}(z+\frac{1}{z})\text{....}(1).$$

The purely geometrical treatment given here is as simple and perhaps simpler than the analytical treatment.*

The equation indicates that the correspondence between the z and the w-planes is not (1, 1), since z takes two distinct values for one of w. The point w is constructed geometrically in the following manner: *Plot the points xepresented by z and $1/z$ and the mid-point of their join gives the point w.*

Consider a circle about the origin, defined by the equation $|z|=r$. If we write $w_1=z$, $w_2=1/z$, and consequently $w=\frac{1}{2}(w_1+w_2)$, we see that as w_1 describes the circle $|z|=r$ in the positive direction, w_2 describes *with the same same angular velocity* the circle $|z|=1/r$ in the opposite direction. This follows from the fact that if we denote

$$\left.\begin{aligned} z&=\rho(\cos\theta+i\sin\theta),\\ \frac{1}{z}&R(\cos\varphi+i\sin\varphi),\end{aligned}\right\}\text{....}(2),$$

the resulting conditions are

$$R=1/\rho,\ \varphi=-\theta\text{....}(3).$$

---

* Holzmueller, *Isogonalen Verwandschaften*, §61, et seq.

If the two circles be described in the sense explained, the point w describes a certain curve which cuts the axes at the points $[\pm\frac{1}{2}(\frac{1}{r}+r), 0][0, \pm(\frac{1}{r}-r)]$ respectively. From the method of description it follows that the curve is symmetrical with respect to the axes. Describe about the origin as center a circle C of radius $\frac{1}{2}(1/r+r)$ and consider for a moment the point M, a point on the locus of the point w, derived from the points F and F' (Fig. 2). The line ML perpendicular to the X-axis is parallel to the line GF, and since ML is parallel to the base of the triangle FGF' and passes through the middle point M of one side FF' of the triangle, therefore it must pass through the middle point N of the other side OF'. Hence it meets the line OF' at the distance $\frac{1}{2}(1/r+r)$ from the center O, and accordingly the point N lies on the circle C. Now

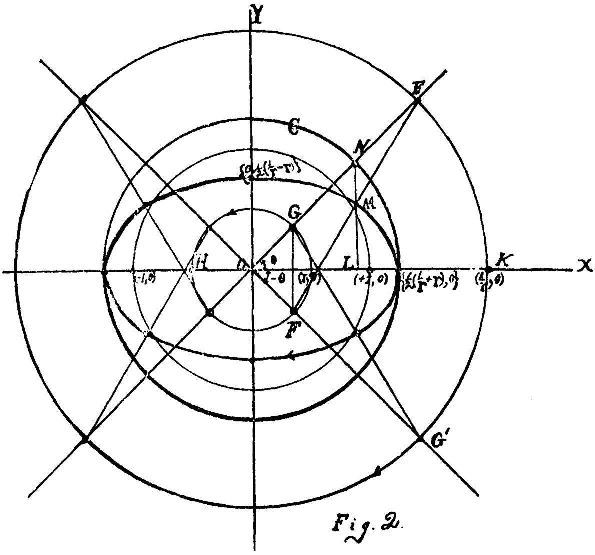

Fig. 2.

$$\frac{NM+ML}{NM}=\frac{NL}{\frac{1}{2}GF}=\frac{ON}{OG}=\frac{\frac{1}{2}(1/r+r)}{r}$$

from which we have

$$NM=\frac{r}{\frac{1}{2}(1/r-r)}ML$$

and substituting we obtain

$$\frac{NL}{ML}=\frac{\frac{1}{2}(1/r+r)}{\frac{1}{2}(1/r-r)}.$$

This proves that the point w describes an ellipse of semi-axes $\frac{1}{2}(1/r+r)$, $\frac{1}{2}(1/r-r)$.* If we have $0<r\leqq 1$, the sense of description is opposite to that of the circle $|z|=r$. If, however, $r=1\dots\infty$, the ellipse and the circle $|z|=r$ are described in the same sense. Thus when G has reached the point $(0, r)$, G' has reached the point $(0, -\frac{1}{r})$ and M has reached the point $0, \frac{1}{2}(r-\frac{1}{r})$, a point below the X-axis; and so in other cases.

---

* C. Smith, *Conic Sections*, §121, page 123.

If z describe the circle KFG', the point w will trace out same ellipse as when the point z described the circle FGH. Since these circles are inverse with respect to the unit circle, the image of which is the segment from $(+1, 0)$ to $(-1, 0)$, we see that the cut for the Riemann surface must be along this segment. The points $(\pm 1, 0)$ are the foci of the ellipses, since $[\frac{1}{2}(\frac{1}{r}+r)]^2-[\frac{1}{2}(\frac{1}{r}-r)]^2=1$. Any point P on the unit circle, considered as a limiting case of both sets of circles, has accordingly two images placed exactly opposite each other with respect to the segment from $(+1, 0)$ to $(-1, 0)$, a direct consequence of the sense of description detailed above. Hence the portion of the upper sheet (upon which we shall map the region of the z-plane exterior to the unit circle) which is below the horizontal axis is joined to the portion of the lower sheet (upon which we shall map the region of the z-plane interior to the unit circle) above the horizontal axis. The junction is made along the cut from $(+1, 0)$ to $(-1, 0)$. The other portions of the two sheets, upper and lower, are also joined. The whole of each ellipse lies entirely in one or the other of the two sheets, since the points $(\pm 1, 0)$ are the foci of every ellipse.

Consider next the map of a half-ray ${}_0P_\infty$ through the origin (Fig. 3). As the point z moves along the half ray $\theta=\alpha$ from 0 to ∞, the point $1/z$ moves along the half-ray $\varphi=-\alpha$ from ∞ to 0. As $|z|$ ranges from 0 to 1, $\frac{1}{|z|}$ ranges from ∞ to 1; and as $|z|$ ranges from 1 to ∞, $\frac{1}{|z|}$ ranges from 1 to 0. Hence the points z and $1/z$ describe along the two half-rays two projective ranges of points.

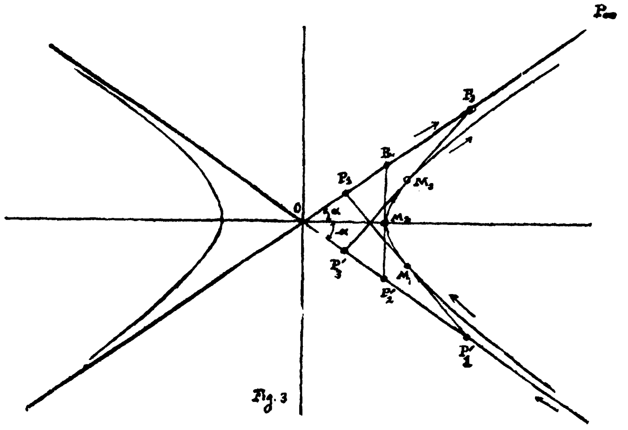

Fig. 3

Since $w=\frac{1}{2}(z+\frac{1}{z})$, the mid-point of any segment PP_1' describes the locus of w. Now the lines PP_1', P_2P_2', ... envelope a conic section which has two points at ∞, one on each half-ray and hence is a hyperbola, to which the half-rays are the asymptotes. When OP_2 is equal to 1, OP_2' is also equal to 1 and M_2 is a vertex of the hyperbola and since OP_2 is distance from the center to the focus, we find that the foci of the hyperbola are $(\pm 1, 0)$. That the system of hyperbolae should be confocal and therefore orthogonal to the former system of ellipses was to be expected, since they were the images of orthogenal loci. The correspondence and sense of description is patent from the drawings, both by means of the lettering and the arrows. For example, let the point z describe the line EOC beginning at D_∞ in the lower left hand corner. As z moves from D_∞ through E to F, w moves in the upper sheet from D_∞'' through E'' to F''; as

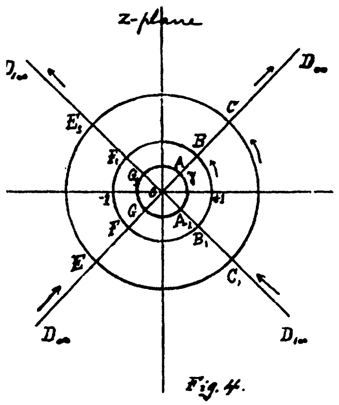

Fig. 4.

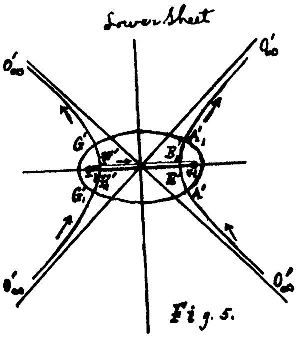

Fig. 5.

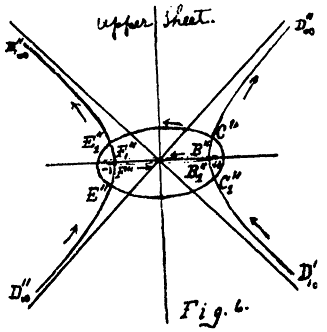

Fig. 6.

z moves from F through G to O, w moves in the lower sheet from F' through G' to O_∞'; as z moves from O through A to B, w moves in the lower sheet from O_∞' lower right hand) through A' to B; and lastly, as z moves from B through C to D_∞, w moves in the upper sheet from B'' through C'' to D_∞''.

THE UNIVERSITY OF CHICAGO, *March, 1903.*

THE GENERAL EUCLIDEAN CONSTRUCTION.

By CHARLES H. SISAM, A. B., Graduate Scholar, Cornell University.

Required to construct, with straight edge and compasses, a length y, given the relation $y=\varphi(a, b, \ldots., l)$; where $a, b, \ldots., l$ represent known lengths and φ is a function involving only rational expressions and square roots.

In order that the construction shall be possible, φ must be homogeneous of first degree. For, let $y_1, a_1, b_1, \ldots., l_1$ be the numerical values $y, a, b, \ldots., l$ for some unit of length z. Then for any other unit of length $(1/k)z$, we must have $ky_1 \equiv \varphi(ka_1, kb_1, \ldots., kl_1)$ for all values of k.

Consider one of the radicals of φ such that the expression under it does not contain any radicals. This expression is composed of a sum of terms of the form $\frac{P}{Q}a^\alpha b^\beta \ldots. l^\lambda$; where $P, Q, \alpha, \beta, \ldots., \lambda$ are integers; and, for all the terms of the expression $\alpha+\beta+\ldots.+\lambda=c$, a constant, which can—by removing from under the radical sign, if necessary, an even power of any of the quantities $a, b, \ldots., l$—be made equal to 2.

On any line, lay off a convenient length $ON \equiv x$. At N erect a perpendicular and on the perpendicular lay off $NA=a$, $NB=b$, etc. Draw $OA, OB, \ldots, OL$.